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IMPROVEMENT OF A PRESSURE GRADIENT METHOD AND ITS APPLICATION TO AN UNSTEADY FLOW PROBLEM

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SUMMARY

The pressure gradient method using velocity components and components of a pressure gradient as dependent variables has been modified to solve incompressible Newtonian fluid flow problems numerically. Applying this modified method to unsteady-state development of flow in a circular cavity shows that, at least for the case of a low Reynolds number flow, relative errors produced by the proposed method are smaller for most time intervals than those produced by the primitive velocity–pressure variable method and by the standard pressure gradient method. Also it is found that the modified and standard pressure gradient methods can be applied to the unsteady circular cavity flow at a moderate Reynolds number of at least up to 200.

KEY WORDS Pressure Gradient Method Unsteady Flow Cavity Flow

INTRODUCTION

The Navier–Stokes equations governing incompressible Newtonian fluids are expressed generally in the form of non-linear second order partial differential equations. Thus, analytical solutions in a closed form, or self-similarity solutions expressible by ordinary differential equations, have been obtained in very limited situations. Therefore, the necessity to simulate the Navier–Stokes equations numerically with boundary conditions and/or initial conditions is increasing to obtain detailed flow information such as velocity fields numerically. The procedure for numerical simulation of the Navier–Stokes equations can be divided into three categories: (1) choice of independent and dependent variables, (2) discretization of the governing equations, (3) numerical solution of the discretized equations. Of these, the second and the third are common to most numerical solution procedures for partial differential equations in space and/or time, and are found elsewhere.[1–5] The remaining item, i.e. how to choose dependent and independent variables, is highly dependent on the form of the governing equations. Although a basic set of dependent variables is known from the construction process of the governing equation(s), some arbitrariness of the selection of dependent variables exists to obtain a higher rate of numerical computation or to obtain better accuracy or stability of numerical computation, because numerical solution processes are very sensitive to the form of the discretized equations, i.e. to the choice of variables.

For the Navier–Stokes equations for incompressible Newtonian fluids in an isothermal flow, pressure and velocity components can be selected as primitive dependent variables. Alternatively, since velocity components can be derived from a stream function (in the case of a two-dimensional flow or an axisymmetric flow) or from a curl of a vector (in the case of a three-dimensional flow), the stream function or the components of the vector can be selected as dependent variables, and

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consequently pressure itself can be obtained from a total differential equation for pressure through the Navier–Stokes equation. On the other hand, since pressure itself is independent of density and temperature for incompressible fluids, and since the Navier–Stokes equation for incompressible Newtonian fluids is governed not directly by pressure itself but by the pressure gradient, it is possible to select a velocity vector and a pressure gradient as unknown dependent variables; such a method is called a pressure gradient method and the possibility of its application to a two-dimensional steady-state flow problem has been briefly discussed;[4] this method results in requiring to develop a new solution procedure or discretizing equations to attain a greater numerical stability.

In this paper, a possibility of improvement of the pressure gradient method for the Navier–Stokes equation is proposed and some features of the pressure gradient method are discussed with numerical examples for an unsteady-state developing flow problem.

ANALYSIS

Formation of a solution procedure

The Navier–Stokes equation for an incompressible isothermal fluid flow, and the continuity equation are expressed as

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\mathbf{V} = -\nabla p + \mu \Delta \mathbf{V} + \mathbf{F}, \tag{1}$$

$$\operatorname{div} \mathbf{V} = 0, \tag{2}$$

respectively. Here the necessary and sufficient condition of compatibility for the pressure p is expressed as

$$\operatorname{curl}(\nabla p) = \mathbf{0}. \tag{3}$$

Especially for a two-dimensional plane flow, two components of equation (1) and one component (in the direction perpendicular to the flow plane) of equation (3) play a role; the remaining components of equation (3) can be satisfied automatically considering the two-dimensionarity.

In the following, dependent variables can be selected as the velocity components and the components of the pressure gradient. Thus equations (1)–(3) are regarded as differential equations for $\mathbf{V}$ and ∇p (instead of $\mathbf{V}$ and p). The pressure p itself at each time can be obtained in terms of ∇p through a total differential equation in space:

$$\mathrm{d}p = \frac{\partial p}{\partial x}\mathrm{d}x + \frac{\partial p}{\partial y}\mathrm{d}y + \frac{\partial p}{\partial z}\mathrm{d}z. \tag{4}$$

Discretization

In the current proposal an arbitrary method of discretization such as a finite difference, an FEM, or a weighted residual for the terms $(\mathrm{D}/\mathrm{D}t)\,\mathbf{V}$ and $\Delta\mathbf{V}$ in equation (1) can be selected. However, since ∇p appears explicitly in equation (1), the following approximation for ∇p at a given time in equation (1) is proposed:

$$(\nabla p)_{\mathbf{X}} = \left(1 - \sum_m \beta_m\right)(\nabla p)_{\mathbf{X}} + \sum_m \beta_m (\nabla p)_{\mathbf{X}+\delta\mathbf{X}_m}, \tag{5}$$

where the subscripts on ∇p denote the location where the gradient ∇p is to be evaluated, the $\delta\mathbf{X}_m$s are the relative position vectors (with respect to the location $\mathbf{X}$) of the neighbouring points surrounding $\mathbf{X}$, and the β_ms are suitable constants (or variables of $\mathbf{X}$ and $\delta\mathbf{X}_m$) such that

$$\sum_m \beta_m < 0, \tag{6}$$

$$\sum_m \beta_m \delta\mathbf{X}_m = \mathbf{0}. \tag{7}$$

Such selection of β_ms is possible if $\mathbf{X}$ lies in the interior of the domain (not on the boundary) as long as points where variables are to be evaluated are moderately distributed in space, to which almost all grid generation would apply. Moreover, if a sufficient number of points surrounding the point $\mathbf{X}$ are available, then the multidimensional normalized vector $(\beta_1, \beta_2, \ldots, \beta_k)/\|(\beta_1, \beta_2, \ldots, \beta_k)\|$ can be determined with a certain arbitrariness (k is the number of points) and a different set of $(\beta_1, \beta_2, \ldots, \beta_k)$ may be assigned to a different component of ∇p. The truncation error included in equation (5) is $O(\max \|\delta\mathbf{X}_m\|^2)$. Methods for discretizing equation (2) and (3) are arbitrary. However, in approximating the partial derivatives through the operators div and curl, it is desirable to retain at least three non-zero terms in order to connect mutual variables at the neighbouring points strongly. For example, it would be better to avoid using a central difference scheme to a first order partial derivative for equally spaced grid points if a finite difference method is applied; instead, if necessary, apply a forward (or backward) difference scheme which can be mixed with a central difference scheme; otherwise, slightly oscillating values with a period of roughly two grid spacings may be superposed on the actual values owing to the lack of direct interaction between two adjacent points.

Although in addition to the spatial estimation as mentioned above it is necessary for unsteady flow problems to specify the time when the spatial estimation is to be made and also to introduce an approximation for partial derivatives with respect to time, no restriction is imposed on what method is selected for the time estimation under the currently proposed pressure gradient method.

AN EXAMPLE OF THE ANALYSIS

Flow configuration

As an example the current proposed modified pressure gradient method is applied to a two-dimensional unsteady circular cavity flow in a horizontal plane; that is, Newtonian fluid enclosed in a circular cavity of radius a is assumed to be initially at rest ($t < 0$) and at $t = 0$ the portion corresponding to a fixed half boundary of radius a suddenly starts to move at a constant speed V in its own curved plane in the counterclockwise direction.

Formulation of the problem

Hereafter for simplicity co-ordinates, pressure, velocity and time are non-dimensionalized with respect to a, ρV^2, V and a/V, respectively. Thus, as usual, a single parameter appears in the equations of motion as a Reynolds number $Re(\equiv \rho a V/\mu)$. To describe the motion, a cylindrical polar co-ordinate system (r, θ), locating its origin at the centre of the cavity, is used with a Cartesian co-ordinate system (x, y) as in usual orientation with the common origin, and without loss of generality the moving boundary is assumed to be $r = 1$, $-\pi < \theta < 0$. Initial conditions at $t = 0$ are expressed as

$$\left.\begin{array}{lll} u = v = 0, & \text{for} & |r| < 1, \\ u = v = 0, & \text{for} & r = 1, 0 < \theta < \pi, \\ u = 0, v = 1 & \text{for} & r = 1, -\pi < \theta < 0, \end{array}\right\} \tag{8}$$

where u and v denote the radial and circumferential components of velocity, respectively. Boundary conditions are

$$u=0, \quad \text{for} \quad r=1, \tag{9}$$

$$v=\begin{cases}0, & \text{for} \quad 0<\theta<\pi, r=1\\ 1, & \text{for} \quad -\pi<\theta<0, r=1.\end{cases} \tag{10}$$

Grid points where a velocity vector $\mathbf{V}$ and a pressure gradient ∇p are to be evaluated consist of the origin and other points (r_i, θ_j), where

$$r_i=\frac{i}{M}, \quad i=1,2,\ldots,M \ (M \text{ a suitable integer}),$$

$$\theta_j=(j-0{\cdot}5)\frac{\pi}{N}, \quad j=-N+1,\ldots,0,1,\ldots,N \quad (N \text{ a suitable integer}),$$

so that the two singular points $(r=1, \theta=0)$ and $(r=1, \theta=\pi)$ are not included in the grid points. Among many possibilities, the components of ∇p at the point (r_i, θ_j) in the interior of the domain are derived from different expressions using a common negative parameter β, i.e.

$$\left(\frac{\partial p}{\partial r}\right)_{i,j}\approx\beta\left(\frac{\partial p}{\partial r}\right)_{i-1,j}+(1-2\beta)\left(\frac{\partial p}{\partial r}\right)_{i,j}+\beta\left(\frac{\partial p}{\partial r}\right)_{i+1,j}, \tag{11}$$

$$\left(\frac{1}{r}\frac{\partial p}{\partial\theta}\right)_{i,j}\approx\beta\left(\frac{1}{r}\frac{\partial p}{\partial\theta}\right)_{i,j-1}+(1-2\beta)\left(\frac{1}{r}\frac{\partial p}{\partial\theta}\right)_{i,j}+\beta\left(\frac{1}{r}\frac{\partial p}{\partial\theta}\right)_{i,j+1}. \tag{12}$$

The pair (i, j) in the subscripts means that the said value followed by the pair is to be evaluated at the point (r_i, θ_j). In the case of $j=-N+1$ and $j=N$, $j-1$ and $j+1$ in equation (12) should be read as N and $-N+1$, respectively. Also in the case of $i=1$, the term $(\partial p/\partial r)_{i-1,j}$ stands for the value $(\partial p/\partial r)$ at $(r=0, \theta=\theta_j)$ and

$$\left(\frac{\partial p}{\partial r}\right)_{(r=0,\theta=\theta_j)}\equiv\left(\frac{\partial p}{\partial x}\right)_0\cos\theta_j+\left(\frac{\partial p}{\partial y}\right)_0\sin\theta_j. \tag{13}$$

Note that the x- and y-components of the vector ∇p at the origin are denoted as $(\partial p/\partial x)_0$ and $(\partial p/\partial y)_0$ respectively and equation (5) does not apply to this case. The components of the vector ∇p at the wall can be obtained through the equations of motion evaluated there. Through equations (11) and (12), applying $\beta=0$ reduces to the standard pressure gradient method. The x- and y-components of the velocity vector at the origin are denoted as U_0 and V_0, respectively. Then r- and θ-components of the velocity at the origin, denoted as u_r and u_θ, become

$$u_r=U_0\cos\theta+V_0\sin\theta, \tag{14}$$

$$u_\theta=-U_0\sin\theta+V_0\cos\theta. \tag{15}$$

Thus equations (1)–(3) can be discretized throughout the interior of the domain as follows:

(i) For the local acceleration at the current time t

$$\frac{\partial}{\partial t}\mathbf{V}(\mathbf{X},t)\approx\frac{1}{\delta t}\left\{\mathbf{V}(\mathbf{X},t+\delta t)-\mathbf{V}(\mathbf{X},t)\right\}, \tag{16}$$

where δt is a time increment.

(ii) All the convected terms are evaluated at time t with suitably selected finite difference approximations.

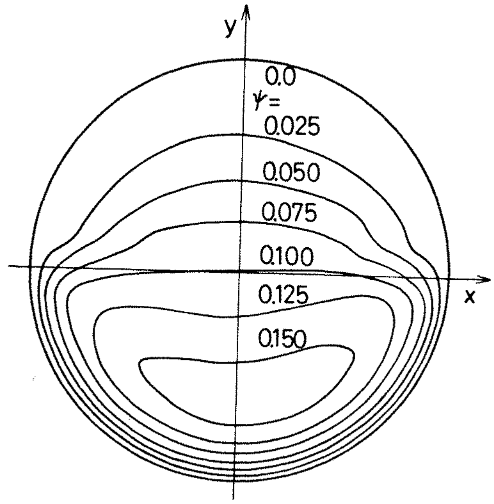

Figure 1. Streamlines in an unsteady flow in a circular cavity at $t = 32/121$ ($Re = 10$)

(iii) All the other terms are evaluated at time $t + \delta t$ also with suitably selected finite difference approximations, including boundary conditions.

Especially at the origin the differential forms of equations (2) and (3) for $\mathbf{V}$ and ∇p are satisfied automatically, considering the component expression such as equations (14) and (15) for $\mathbf{V}$ and a similar one for ∇p. Therefore, the continuity of either $\mathbf{V}$ or ∇p at the origin should be supplemented instead of equations (2) and (3) at the origin; this should be evaluated at time $t + \delta t$.

Such a discretization constitutes a set of full implicit simultaneous linear equations for the unknowns at time $t + \delta t$; this can be solved with the initial conditions.

Numerical results

Figures 1 and 2 show patterns of streamlines and isobars for $Re = 10$, $t = 32/121$ ($\beta = -0{\cdot}5$,

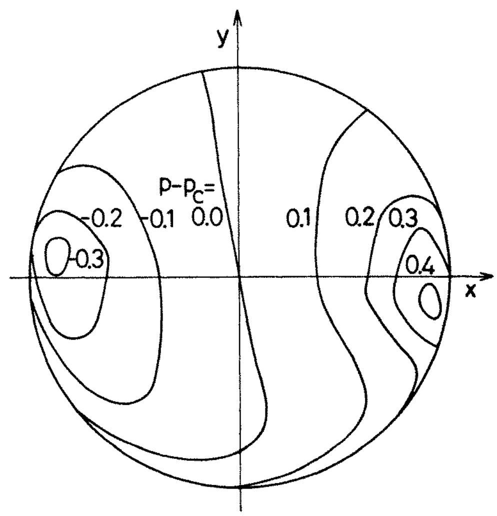

Figure 2. Isobars in an unsteady flow in a circular cavity at $t = 32/121$ ($Re = 10$)

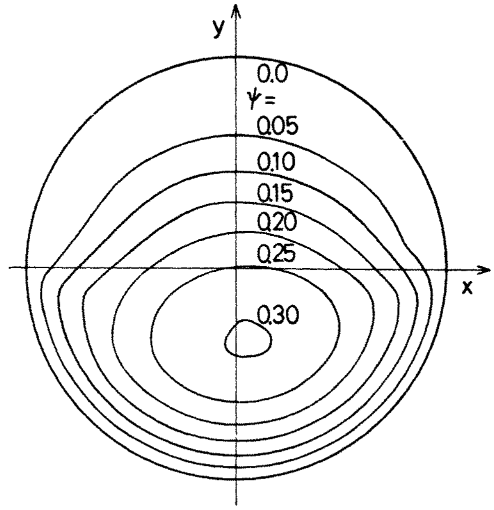

Figure 3. Streamlines at a sufficiently large time in a circular cavity, which correspond to those in a steady flow ($Re = 10$)

$M = 11$, $N = 10$, $\delta t = 1/121$), where stream functions are calculated directly by integrating the velocity component, and the pressure itself is due to equation (4). Figures 3 and 4 show patterns of streamlines and isobars for $Re = 10$, $t = \infty$ ($\beta = -0{\cdot}5$, $M = 11$, $N = 10$), which correspond to the steady-state flow patterns (obtained at sufficiently large time).

DISCUSSION

Comparison with other method

In the case of a finite value of Re, non-linear convective acceleration terms do not vanish and it is extremely hard to get an exact analytical solution. Thus the applicability of equation (5)

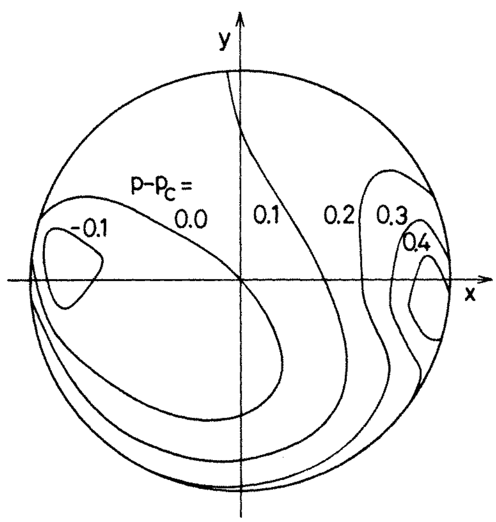

Figure 4. Isobars at a sufficiently large time in a circular cavity, which correspond to those in a steady flow ($Re = 10$)

can be compared for sufficiently small values of Re, using an asymptotic analytical solution for $Re \to 0$, where the stream function ψ and the pressure p can be expressed as

$$\psi = -\tfrac{1}{4}(r^2-1) + \frac{r^2-1}{2\pi}\tan^{-1}\left(\frac{2r\sin\theta}{1-r^2}\right)$$
$$+ \sum_{m=1}^{\infty} \frac{1}{\beta_{0m}^2 J_0(\beta_{0m})}\{J_0(\beta_{0m}r) - J_0(\beta_{0m})\}\exp(-\beta_{0m}^2 t/Re)$$
$$+ \sum_{n=1,3,5,\ldots} \frac{4}{n\pi}\left[\sum_{m=1}^{\infty} \frac{1}{\beta_{nm}^2 J_n(\beta_{nm})}\{r^n J_n(\beta_{nm}) - J_n(\beta_{nm}r)\}\exp(-\beta_{nm}^2 t/Re)\right]\sin(n\theta), \quad (t>0), \tag{17}$$

$$p - p_c = \frac{1}{Re}\sum_{n=1,3,5,\ldots} \frac{4}{n\pi}\left\{n+1+\sum_{m=1}^{\infty}\exp(-\beta_{nm}^2 t/Re)\right\} r^n\cos(n\theta), \quad (t>0), \tag{18}$$

where $\beta_{nm}(n=0, 1, 3, 5, \ldots)$ is the mth positive zero of the Bessel function $J_{n+1}(x)$ and p_c is the pressure at the centre of the cavity. Using equation (18), numerical errors mainly due to discretization can be estimated and the error behaviour with time can be compared among methods (a finite difference method where $\mathbf{V}$ and p are supposed to be dependent variables ($\mathbf{V}-p$ method), a finite difference standard pressure gradient method where $\mathbf{V}$ and ∇p are supposed to be dependent variables ($\beta = 0$), and a finite difference modified pressure gradient method where $\mathbf{V}$ and ∇p are supposed to be dependent variables ($\beta < 0$)); these are shown in Figures 5 and 6, where E_p and $E_{\nabla p}$ denote measures of pressure errors and pressure gradient errors, respectively, and are defined as

$$E_p = \frac{1}{n}\sum\{(p-p_c)_{\text{num.}} - (p-p_c)_{\text{anal.}}\}^2/\{(p-p_c)_{\text{anal.}}\}^2, \tag{19}$$

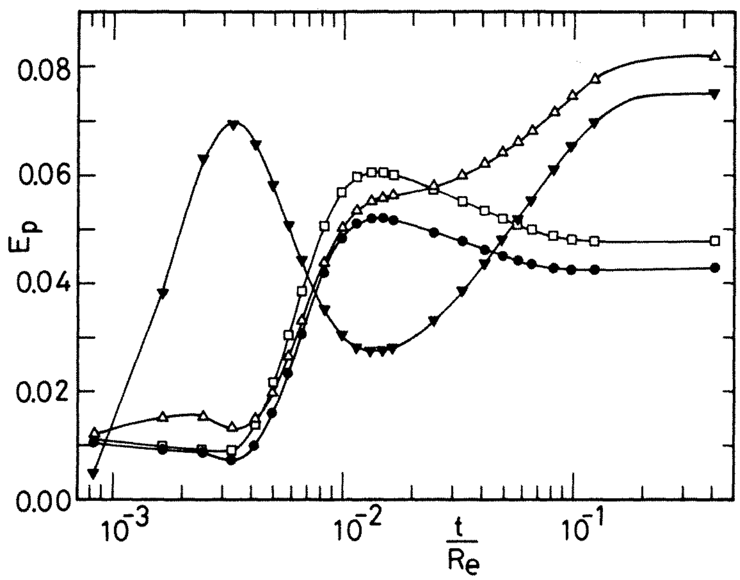

Figure 5. Error behaviour of E_p with time for different methods possessing the same parameters of common values ($Re = 0{\cdot}001$, $\delta t = 10^{-4}/121$, $M = 11$, $N = 10$). ▼: a $\mathbf{V}-p$ method, △: $\beta = 0$ (a standard pressure gradient method), ●: $\beta = -0{\cdot}5$ (a modified pressure gradient method), □: $\beta = -1{\cdot}0$ (a modified pressure gradient method)

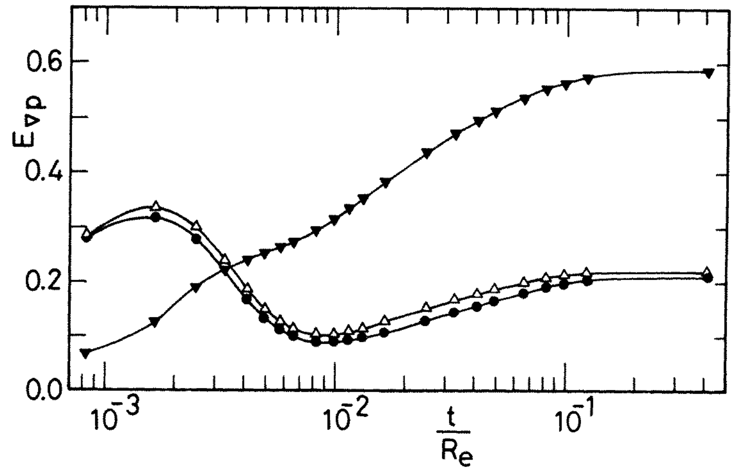

Figure 6. Error behaviour of $E_{\nabla p}$ with time for different methods possessing the same parameters of common values ($Re = 0{\cdot}001$, $\delta t = 10^{-4}/121$, $M = 11$, $N = 10$). ▼: a V–p method, △: $\beta = 0$ (a standard pressure gradient method), ●: $\beta = -0{\cdot}5$ (a modified pressure gradient method). Since the trend of data for $\beta = -1{\cdot}0$ is approximately equal to that for $\beta = -0{\cdot}5$, data for $\beta = -1{\cdot}0$ have been omitted

$$E_{\nabla p} = \frac{1}{n}\sum |(\nabla p)_{\text{num.}} - (\nabla p)_{\text{anal.}}|^2 / |(\nabla p)_{\text{anal.}}|^2, \tag{20}$$

where n is the total number of points to be evaluated, and the subscripts num. and anal. mean 'obtained numerically', and 'obtained analytically', respectively. In Figures 5 and 6, all the discretizing methods possess the same spatial division and the same time increment, and for the estimation of equation (19), pressure is evaluated at the grid points over three concentric circles ($r = 10/11$, $7/11$, $4/11$) ($n = 60$), and for the estimation of equation (20) pressure gradients are evaluated at points over three concentric circles ($r = 19/22$, $13/22$, $7/22$) ($n = 60$). As far as the current spatial division and time increment at a specified small value of Re are concerned, as in Figures 5 and 6, the current modified pressure gradient method possessing a parameter of $\beta = -0{\cdot}5$ produces better accuracy for most time intervals.

Stability criteria for numerical time integration

Under the current initial and boundary conditions, the transient solution would be expected to approach the steady-state solution so long as Re is not sufficiently large. Thus it is desirable that a numerical transient solution will approach a steady state after a sufficient time elapses. Such a character can be retained not only for small values of Re as shown in Figures 3 and 4, but also even for $Re = 200$ under the current pressure gradient method ($\beta = -0{\cdot}25$, $\delta t = 10/121$, $M = 11$, $N = 10$) and also under the standard pressure gradient method ($\beta = 0$, $\delta t = 10/121$, $M = 11$, $N = 10$). This means that the pressure gradient method can also be used to find a steady-state solution through a transient solution. The optimum value of β or in general $\sum \beta_m$ would depend on Re, δt, and also grid spacing.

CONCLUSION

A modified pressure gradient method has been proposed and applied to an unsteady-state developing flow; this shows that the current method using a suitable value of a parameter can give better accuracy at least at a small value of Re than the other compared methods and that numerical time integration can produce a steady-state solution through a transient solution even for $Re = 200$.

NOTATION

a	radius of a cavity
E_p	relative errors of pressure (defined in equation (19))
$E_{\nabla p}$	relative errors of a pressure gradient (defined in equation (20))
$\mathbf{F}$	external body force
p	pressure
r	radial co-ordinate in a cylindrical co-ordinate system
Re	Reynolds number $\equiv \rho a V/\mu$
t	time
u	radial velocity component
V	circumferential velocity of a moving wall
$\mathbf{V}$	velocity vector
v	circumferential velocity component
$\mathbf{X}$	location vector
x	co-ordinate in a Cartesian co-ordinate system
y	co-ordinate in a Cartesian co-ordinate system
β	parameter introduced in equation (11)
β_m	parameter introduced in equation (5)
θ	tangential co-ordinate in a cylindrical co-ordinate system
μ	viscosity of fluid
ρ	density of fluid
ψ	stream function
$\mathrm{D}/\mathrm{D}t$	material time derivative operator
Δ	Laplacian operator
∇	gradient operator

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